

# ASYMPTOTICS FOR A RESONANCE-COUNTING FUNCTION FOR POTENTIAL SCATTERING ON CYLINDERS

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**ABSTRACT.** We study certain resonance-counting functions for potential scattering on infinite cylinders or half-cylinders. Under certain conditions on the potential, we obtain asymptotics of the counting functions, with an explicit formula for the constant appearing in the leading term.

## 1. INTRODUCTION

We study potential scattering on infinite cylinders and half-cylinders. In particular, we give some sharp upper bounds and some asymptotics for resonance-counting functions in this setting.

Let  $X = (-\infty, \infty) \times Y$ , or  $[0, \infty) \times Y$ , where  $Y$  is a smooth compact, connected manifold, with or without boundary. We consider the product metric

$$(dx)^2 + g_Y,$$

where  $g_Y$  is a smooth metric on  $Y$ . Let  $\Delta$  be the Laplacian on  $X$ , with Dirichlet or Neumann boundary conditions if  $X$  has a boundary. We consider operators  $\Delta + V$ , where  $V \in L_{\text{comp}}^\infty(X; \mathbb{C})$ .

Let  $\Delta_Y$  be the Laplacian on  $Y$ , with boundary conditions if necessary, and let  $\{\sigma_j^2\}$ ,  $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2 \leq \dots$  be the set of all eigenvalues of  $\Delta_Y$ , repeated according to their multiplicity, and let  $\nu_1^2 < \nu_2^2 < \nu_3^2 < \dots$  be the *distinct* eigenvalues of  $\Delta_Y$ . Then the resolvent of the Laplacian  $\Delta$  on  $X$ , or of  $\Delta + V$ , for  $V \in L_{\text{comp}}^\infty(X)$ , has a meromorphic continuation to the Riemann surface  $\hat{Z}$  on which  $r_j(z) = (z - \nu_j^2)^{1/2}$  is a single-valued function for all  $j$  [11, 13]. Thus the resonances, poles of the meromorphic continuation of the resolvent, are points in  $\hat{Z}$ . In many settings, resonances correspond to waves which eventually decay. Additionally, they are in many ways analogous to eigenvalues. Because of this, they have been widely studied—see [16, 19, 20] for an introduction to resonances and for further references.

Here we study a simple case of scattering on manifolds with infinite cylindrical ends. The spectral and scattering theory of such manifolds exhibits some characteristics one expects both from one-dimensional scattering and from  $n$ -dimensional spectral theory (if  $\dim X = n$ ). The resonance-counting functions we consider here demonstrate the one-dimensional nature of the scattering. Evidence of the

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$n$ -dimensional nature can be seen, for example, in the Weyl asymptotics or in the maximal rate of growth of the eigenvalue-counting function [6, 14]. It also appears in some resonance-counting functions, e.g. [4].

For  $z \in \mathbb{C} \setminus [\nu_1^2, \infty)$ ,  $R_V(z) = (\Delta + V - z)^{-1}$  is bounded on  $L^2(X)$  except, perhaps, for a (perhaps infinite, if  $V$  is complex-valued) set of points corresponding to eigenvalues. Considered as a map from  $L^2_{\text{comp}}(X)$  to  $H^2_{\text{loc}}(X)$ ,  $R_V$  has a meromorphic continuation to the Riemann surface  $\hat{Z}$  described earlier. Let  $r_j(z) = (z - \nu_j^2)^{1/2}$  and let  $\tilde{r}_k(z) = r_j(z)$  if  $\sigma_k^2 = \nu_j^2$ .

**Theorem 1.1.** *Let  $X = (-\infty, \infty) \times Y$  and let  $V \in L^\infty_{\text{comp}}(X; \mathbb{C})$ . Fix a sheet of  $\hat{Z}$ , and suppose that  $\text{Im } r_{j_0}(z) < 0$  on this sheet. Then, there is a constant  $c_{V,\mathcal{E}} \geq 0$  such that for any  $\alpha > 0$ ,*

$$\#\{z_k : z_k \text{ is a pole of } R_V(z) \text{ on this sheet},$$

$$|r_{j_0}(z_k)| < r, \text{ Im } r_{j_0}(z_k) < -\alpha\} = c_{V,\mathcal{E}}r + o_\alpha(r)$$

The constant  $c_{V,\mathcal{E}}$  depends on the potential  $V$  and the sheet (indicated by  $\mathcal{E}$ ). Moreover,

$$c_{V,\mathcal{E}} \leq \frac{2}{\pi} \left( \max_{(x,y), (x',y') \in \text{supp } V} |x - x'| \right) \#\{l : \text{Im } \tilde{r}_l(z) < 0 \text{ when } z \text{ lies on this sheet}\}.$$

Here, as everywhere, we count resonances with multiplicities. The error term  $o_\alpha(r)$  depends on  $V$  and on the sheet as well as on  $\alpha$ , of course.

We remark that this bound on the constant  $c_{V,\mathcal{E}}$  is sharp, as can easily be seen by considering a potential that depends only on  $x$ , and using the results of [17] or [10] for potential scattering on the line.

Although Theorem 1.1 gives, in some sense, asymptotics of a resonance-counting function, it does not give meaningful lower bounds on the size of  $c_{V,\mathcal{E}}$ . In some settings we are able to actually determine  $c_{V,\mathcal{E}}$ , but we need some additional conditions on  $V$ .

Let  $\{\phi_j\}$  be an orthonormal set of eigenfunctions of  $\Delta_Y$  associated with  $\sigma_j^2$ . By translating if necessary, we can, in the case of the full cylinder, arrange that for some  $b \in \mathbb{R}$ , the support of  $V$  is contained in  $[-b, b] \times Y$ , but is not contained in the product of any smaller interval with  $Y$ .

**Theorem 1.2.** *Let  $X = (-\infty, \infty) \times Y$  and suppose that the support of  $V$  is contained in  $[-b, b] \times Y$  and the interval  $[-b, b]$  cannot be replaced by a smaller one. Restrict ourselves to a sheet of  $\hat{Z}$  with  $\text{Im } r_j(z) < 0$  if and only if  $j = j_0$ . Suppose that  $\nu_{j_0}^2$  is a simple eigenvalue of  $\Delta_Y$ , with  $\nu_{j_0}^2 = \sigma_{l_0}^2$ , and that*

(1)

$$C|V_{l_0 l_0}(x)| = C \left| \int_Y V(x, y) |\phi_{l_0}(y)|^2 d\text{vol}_Y \right| \geq |V(x, y)|, \text{ for } |x - b| < \epsilon, |x + b| < \epsilon$$

for some  $C, \epsilon > 0$ . Then, for any  $\alpha > 0$ ,

$$\begin{aligned} \#\{z_k : z_k \text{ is a pole of } R_V(z) \text{ on this sheet}, |r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\} \\ = \frac{4}{\pi} br + o_\alpha(r). \end{aligned}$$

In Section 4 we give an example of a nontrivial complex-valued potential for which (1) is not satisfied and for which the conclusion of the theorem does not hold. Moreover, for this potential  $c_{V,\varepsilon} = 0$  for at least one (non-physical) sheet. This gives an example of some behaviour which is even asymptotically truly different from that demonstrated by scattering by the family of potentials  $V(x)$ . Moreover, this means that potential scattering on cylinders provides an example of a setting in which even the order of growth of a resonance-counting function may vary depending on the potential.

In Section 4 we prove a theorem which gives another situation in which we can determine  $c_{V,\varepsilon}$ . In Section 5 we give some analogous results for potential scattering on half-cylinders.

Scattering on cylinders bears some resemblance to potential scattering on the line. On the line, the distribution of resonances has been studied in [10, 15, 17]. The complicated nature of  $\hat{Z}$  makes more difficult the question of bounding the number of resonances in the cylindrical end setting. Earlier results on resonances for manifolds with cylindrical ends include [2, 4, 8, 9], and references. For general scattering theory on manifolds with cylindrical ends, references include [11, 13].

## 2. PRELIMINARIES AND NOTATION

Let  $r_j(z) = (z - \nu_j^2)^{1/2}$  and identify the physical sheet of  $\hat{Z}$  as being the part of  $\hat{Z}$  on which  $\operatorname{Im} r_j(z) > 0$  for all  $j$  and all  $z$  and on which  $R_V(z)$  is bounded on  $L^2(X)$  for all but a discrete set of  $z$ . Other sheets will be identified, when necessary, by indicating for which values of  $j$   $\operatorname{Im} r_j(z) < 0$ . Each sheet can be identified with  $\mathbb{C} \setminus [\nu_1^2, \infty)$ . With this language, there are points in  $\hat{Z}$  which belong to no sheet but which belong to the boundary of the closure of two sheets, and the ramification points, which correspond to  $\{\nu_j^2\}$  and belong to the closure of four sheets (except for ramification points corresponding to  $\nu_1^2$ ). We note that sheets that meet the physical sheet are characterized by the existence of a  $J \in \mathbb{N}$  such that

$$\operatorname{Im} r_j(z) < 0 \text{ for all } z \text{ on that sheet if and only if } j \leq J.$$

We can associate to a fixed sheet of  $\hat{Z}$  a set  $\mathcal{E} \subset \mathbb{N}$ ,

$$\mathcal{E} = \{j : \operatorname{Im} r_j(z) < 0 \text{ on this sheet}\}.$$

We shall call  $\mathcal{E}$  the *labeling set*. Let

$$\tilde{\mathcal{E}} = \{l \in \mathbb{N} : \sigma_l^2 = \nu_j^2 \text{ for some } j \in \mathcal{E}\}.$$

Let  $\{\phi_j\}$  be an orthonormal set of eigenfunctions of  $\Delta_Y$  associated with  $\{\sigma_j^2\}$ .

In general, we shall use  $z$  to stand for a point in  $\hat{Z}$  and  $\Pi(z)$  to represent its projection to  $\mathbb{C}$ . For  $w \in \mathbb{R}^m$ ,  $\langle w \rangle = (1 + |w|^2)^{1/2}$ . We will denote by  $C$  a constant whose value may change from line to line.

Next we recall some results and language of complex analysis, e.g. [12], and recall a theorem we shall need on the distribution of zeros of functions which are “good” in a half-plane.

We shall often work with functions that are holomorphic not in the whole plane but are holomorphic within an angle  $(\theta_1, \theta_2)$ . A function  $F$  holomorphic in an angle  $(\theta_1, \theta_2)$  is of order  $\rho$  there if

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln \ln(\sup_{\theta \in (\theta_1, \theta_2)} |F(re^{i\theta})|)}{r} = \rho.$$

A function of order  $\rho$  in the angle  $(\theta_1, \theta_2)$  is of type  $\tau$  there if

$$\overline{\lim}_{r \rightarrow \infty} \frac{\ln \sup_{\theta \in (\theta_1, \theta_2)} |F(re^{i\theta})|}{r^\rho} = \tau.$$

A function of order 1 and type  $\tau < \infty$  (in an angle  $(\theta_1, \theta_2)$ ) is said to be of exponential type there. Of course,  $\rho$  and  $\tau$  can depend on  $\theta_1$  and  $\theta_2$ .

The indicator of a function  $F$  holomorphic in an angle  $\theta_1 \leq \arg \zeta \leq \theta_2$  and of order  $\rho$  is

$$h_F(\theta) = \overline{\lim}_{r \rightarrow \infty} \frac{\ln |F(re^{i\theta})|}{r^\rho}.$$

A function  $F$  is of completely regular growth within the angle  $(\theta_1, \theta_2)$  if

$$\lim_{\substack{r \rightarrow \infty \\ r \notin E}} \frac{\ln |F(re^{i\theta})|}{r^\rho} = h_F(\theta)$$

where the set  $E \subset \mathbb{R}_+$  is of zero relative measure and the convergence is uniform for  $\theta \in (\theta_1, \theta_2)$ .

We shall abuse notation slightly and also use the language above for a function that is holomorphic for  $\theta_1 \leq \arg \zeta \leq \theta_2$  and  $\zeta$  outside of a compact set.

For a function  $f$  defined in the lower half plane, let  $n_{f-}(r)$  be the number of zeros of  $f$ , counted with multiplicity, that lie in the lower half-plane and have norm less than  $r$ .

**Theorem 2.1.** *Suppose  $f(\zeta)$  is holomorphic in the closed lower half plane  $\text{Im } \zeta \leq 0$ ,*

$$|f(\zeta)| \leq C e^{C|\zeta|}$$

*there,  $f(0) = 1$ ,*

$$\left| \int_{-\infty}^{\infty} \frac{d[\arg f(t)]}{dt} dt \right| < \infty$$

*and*

$$\left| \int_{-\infty}^{\infty} \frac{\ln |f(t)|}{1+t^2} dt \right| < \infty.$$

Then

$$\lim_{r \rightarrow \infty} \frac{n_{f-}(r)}{r} = \frac{1}{2\pi} \int_{\pi}^{2\pi} h_f(\varphi) d\varphi.$$

The proof of this theorem can be found in [5]. It is an adaptation of arguments of [12, Chapter III, Section 2] and [12, Theorem 3, Chapter III, Section 3].

We note, moreover, that the assumptions of Theorem 2.1 mean that  $f$  is a function of completely regular growth in the lower half-plane and that  $h_f(\theta) = c_f |\sin \theta|$  for  $\pi < \theta < 2\pi$ .

### 3. PROOF OF THEOREM 1.1

As in [10], here we find a matrix  $B$  so that the poles of the resolvent in the region in question are included in the zeros of  $\det(I + B)$ . We study the properties of the matrix  $B$ , and then apply Theorem 2.1. Recall that here  $X = (-\infty, \infty) \times Y$ .

Let

$$(2) \quad R_0(z) = (\Delta - z)^{-1} = \sum_{j=1}^{\infty} \frac{i}{2r_j(z)} e^{i|x-x'|r_j(z)} \sum_{\sigma_l^2 = \nu_j^2} \phi_l(y) \overline{\phi}_l(y').$$

Then

$$(\Delta + V - z)R_0(z) = I + VR_0(z).$$

Since  $R_0(z)$  has no null space, away from the ramification points of  $\hat{Z}$ ,  $R_V(z)$  has a pole if and only if  $I + VR_0(z)$  has nontrivial null space (and the multiplicities agree).

If  $\mathcal{E} \subset \mathbb{N}$  is a finite set, define  $w_{\mathcal{E}} : \hat{Z} \rightarrow \hat{Z}$  as follows. To  $z$  we may associate the set of square roots  $\{r_j(z)\}$ . Then  $w_{\mathcal{E}}(z)$  may be determined by saying it is the element of  $\hat{Z}$  associated to the set  $\{r_j(w_{\mathcal{E}}(z))\}$ , with

$$r_j(w_{\mathcal{E}}(z)) = \begin{cases} -r_j(z), & \text{if } j \in \mathcal{E} \\ r_j(z), & \text{if } j \notin \mathcal{E}. \end{cases}$$

Suppose we now restrict ourselves to consider only  $z$  lying on the sheet with

$$\operatorname{Im} r_j(z) < 0 \text{ if and only if } j \in \mathcal{E}.$$

Then  $w_{\mathcal{E}}(z)$  lies in the physical sheet. Moreover,

$$(3) \quad \begin{aligned} I + VR_0(z) &= (I + VR_0(w_{\mathcal{E}}(z))) \left[ I + [I + VR_0(w_{\mathcal{E}}(z))]^{-1} V [R_0(z) - R_0(w_{\mathcal{E}}(z))] \right] \\ &= (I + VR_0(w_{\mathcal{E}}(z))) \left[ I + [I + VR_0(w_{\mathcal{E}}(z))]^{-1} A_1(z) \right] \end{aligned}$$

where  $A_1(z)$  has Schwartz kernel

$$V(x, y) \sum_{l \in \tilde{\mathcal{E}}} \frac{i}{2\tilde{r}_l(z)} (e^{i\tilde{r}_l(z)(x-x')} + e^{-i\tilde{r}_l(z)(x-x')}) \phi_l(y) \overline{\phi}_l(y').$$

If  $|\operatorname{Im} \Pi(w_{\mathcal{E}}(z))| > \|\operatorname{Im} V(x, y)\|_{L^\infty}$ , then  $I + VR_0(w_{\mathcal{E}}(z))$  is invertible. If we restrict ourselves to such  $z$ , then, the poles of the resolvent of  $\Delta + V$  are given by the zeros of

$$\det(I + A_2(z)),$$

where  $A_2(z)$  is

$$A_2(z) = \sum_{l \in \tilde{\mathcal{E}}} \frac{i}{2\tilde{r}_l(z)} (\varphi_{l,+} \otimes \Psi_{l,-} + \varphi_{l,-} \otimes \Psi_{l,+}),$$

with

$$\begin{aligned} \Phi_{l\pm}(x, y, z) &= e^{\pm i\tilde{r}_l(z)x} \phi_l(y) \\ \varphi_{l\pm}(x, y, z) &= \left( (I + VR_0(w_{\mathcal{E}}(z)))^{-1} (V\Phi_{l\pm})(\bullet, z) \right) (x, y) \\ \Psi_{l,\pm}(x, y, z) &= e^{\pm i\tilde{r}_l(z)x} \overline{\phi_l(y)}. \end{aligned}$$

Here we use the notation

$$(f \otimes g)h(x, y) = f(x, y) \int_X g(x', y') h(x', y') d\operatorname{vol}_X.$$

One can then see that the zeros of  $I + A_2(z)$  are the same as the zeros of  $I + A_2(z)\chi$ , where  $\chi \in L_{\text{comp}}^\infty(X)$  is one on the support of  $V$ . The zeros of  $I + A_2(z)\chi$  are the same as the zeros of  $\det(I + B(z))$ , where

$$(4) \quad B(z) = \begin{pmatrix} B_{+-}(z) & B_{--}(z) \\ B_{++}(z) & B_{-+}(z) \end{pmatrix},$$

$B_{+\pm} = (b_{+\pm lj})_{l,j \in \tilde{\mathcal{E}}}$ ,  $B_{-\pm} = (b_{-\pm lj})_{l,j \in \tilde{\mathcal{E}}}$ , and

$$(5) \quad \begin{aligned} b_{+\mp lj}(z) &= \frac{i}{2\tilde{r}_l(z)} \int_X \varphi_{j+}(x, y, z) \chi(x, y) \Psi_{l\mp}(x, y, z) d\operatorname{vol}_X, \\ b_{-\mp lj}(z) &= \frac{i}{2\tilde{r}_l(z)} \int_X \varphi_{j-}(x, y, z) \chi(x, y) \Psi_{l\mp}(x, y, z) d\operatorname{vol}_X. \end{aligned}$$

We shall first obtain upper bounds on the entries in the matrix  $B$ , and thus on  $\det(I + B(z))$ . To do so, we will use the following lemma.

**Lemma 3.1.** *Let  $f_\pm(x, z) = e^{\pm i\tilde{r}_j(z)x}$ , and let  $\chi_1, \chi_2 \in C_c^\infty(X)$ . If  $z$  lies on the physical sheet of  $\hat{Z}$  and  $\operatorname{Im} \tilde{r}_j(z) = t_0 > 0$ , then*

$$\left\| \chi_1 \frac{1}{f_\pm} R_0(z) f_\pm \chi_2 \right\|_{L^2(X) \rightarrow L^2(X)} \leq \frac{C}{|\operatorname{Re} \tilde{r}_j(z)|^{7/12}}$$

when  $|\tilde{r}_j(z)|$  is sufficiently large. Moreover, for  $\operatorname{Im} \tilde{r}_j(z) \geq t_0 > 0$ ,

$$\left\| \chi_1 \frac{1}{f_\pm} R_0(z) f_\pm \chi_2 \right\|_{L^2(X) \rightarrow L^2(X)} \leq \frac{C}{|\tilde{r}_j(z)|^{5/12}}$$

when  $|\tilde{r}_j(z)|$  is sufficiently large.

*Proof.* Without loss of generality we can assume  $\chi_1$  and  $\chi_2$  are independent of  $y$  and thus it suffices to consider, for  $l \in \mathbb{N}$ ,

$$\left\| \chi_1 \frac{1}{f_\pm} R_{0l}(z) f_\pm \chi_2 \right\|_{L^2(X) \rightarrow L^2(X)}$$

where  $R_{0l}(z)$  has Schwartz kernel

$$\frac{i}{2\tilde{r}_l(z)} e^{i\tilde{r}_l(z)|x-x'|} \phi_l(y) \overline{\phi}_l(y').$$

The Schwartz kernel of  $(f_\pm)^{-1} R_{0l}(z) f_\pm$  is

$$K_{l\pm}(x, y, x', y', z) = \begin{cases} \frac{i}{2\tilde{r}_l(z)} e^{i(\tilde{r}_l(z) \mp \tilde{r}_j(z))(x-x')} \phi_l(y) \overline{\phi}_l(y') \chi_1(x) \chi_2(x'), & \text{if } x > x' \\ \frac{i}{2\tilde{r}_l(z)} e^{i(-\tilde{r}_l(z) \mp \tilde{r}_j(z))(x-x')} \phi_l(y) \overline{\phi}_l(y') \chi_1(x) \chi_2(x'), & \text{if } x < x'. \end{cases}$$

We shall show that when  $\operatorname{Im} \tilde{r}_j(z) = t_0$

$$\int_X \int_X |K_{l\pm}(x, y, x', y', z)|^2 d\operatorname{vol}_X d\operatorname{vol}_X \leq \frac{C}{|\operatorname{Re} \tilde{r}_j(z)|^{7/6}},$$

with constant  $C$  independent of  $l$ , which will prove the first part of the lemma.

First, notice that on the support of  $\chi_1(x)\chi_2(x')$ , the exponential function in  $K_{l\pm}$  is bounded independent of  $l$ . This is because  $\operatorname{Im} \tilde{r}_l(z) > 0$  and  $|\operatorname{Im} \tilde{r}_j(z)(x-x')|$  is bounded for  $x \in \operatorname{supp} \chi_1$ ,  $x' \in \operatorname{supp} \chi_2$ . Thus,

$$(6) \quad \|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|\tilde{r}_l(z)|^2}.$$

When  $\tilde{r}_l \neq \tilde{r}_j$ , we may integrate by parts to see that

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|\operatorname{Im}(\tilde{r}_j(z) - \tilde{r}_l(z))|} \frac{1}{|\tilde{r}_l(z)|^2}$$

so that

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|\tilde{r}_l(z)|^2} \min(1, (|\operatorname{Im}(\tilde{r}_j(z) - \tilde{r}_l(z))|)^{-1}).$$

Let  $\tilde{r}_j = s + it_0$ . Then if  $\tilde{r}_l(z) = u + iv$ , a computation shows that, with  $g = \sigma_j^2 + s^2 - t_0^2 - \sigma_l^2$ ,  $u^2 = \frac{1}{2}(g + \sqrt{g^2 + 4s^2t_0^2})$ , and  $v^2 = \frac{1}{2}(-g + \sqrt{g^2 + 4s^2t_0^2})$ . If  $g \leq (|s|t_0)^{7/6}$ , then

$$(7) \quad \begin{aligned} v^2 &\geq \frac{1}{2} \left( -(|s|t_0)^{7/6} + \sqrt{(|s|t_0)^{7/3} + 4(|s|t_0)^2} \right) \\ &= (|s|t_0)^{5/6} + O((|s|t_0)^{1/2}). \end{aligned}$$

Then

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|\tilde{r}_l(z)|^2 |v - t_0|} \leq \frac{C}{(|s|t_0)^{5/6} (|s|t_0)^{5/12}} \leq \frac{C}{|s|^{5/4}}$$

when  $|s|$  is sufficiently large and  $\operatorname{Im} \tilde{r}_j(z) = t_0$ .

If, on the other hand,  $g \geq (|s|t_0)^{7/6}$ , then we use

$$(8) \quad u^2 = \frac{1}{2}(g + (\sqrt{g^2 + 4(st_0)^2})) \geq g \geq (|s|t_0)^{7/6}$$

and

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|\tilde{r}_l(z)|^2} \leq \frac{C}{u^2} \leq \frac{C}{(|s|t_0)^{7/6}}.$$

This finishes the proof of the first part of the lemma.

To prove the second part of the lemma, first notice that if  $\tilde{r}_j(z) = s + it$  and  $|s| < 1$ , then  $\frac{1}{|\tilde{r}_l(z)|^2} \leq \frac{C}{t^2}$  when  $t$  is sufficiently large, so that

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{t^2}$$

in this region. On the other hand, if  $|s| \geq 1$ , the inequalities (6), (7) and (8) together show that when  $t \geq t_0$ ,

$$\|K_{l\pm}(z)\|_{L^2}^2 \leq \frac{C}{|s+it|^{5/6}}.$$

□

Fix  $j_0 \in \mathcal{E}$ . We shall eventually use  $k = r_{j_0}(z)$  to identify our fixed sheet of  $\hat{Z}$  (corresponding to  $\mathcal{E}$ ) with the lower half plane. However, we shall continue to use  $z$  as a coordinate as well, when it is more convenient. In any case, we restrict ourselves to one fixed sheet.

**Lemma 3.2.** *Fix a sheet of  $\hat{Z}$  with corresponding labeling set  $\mathcal{E}$  and let  $j_0 \in \mathcal{E}$ ,  $l, j \in \tilde{\mathcal{E}}$ . If  $-\text{Im } r_{j_0}(z) \geq \alpha > 0$ , then for  $|r_{j_0}(z)|$  sufficiently large (depending on  $\alpha$ ),*

$$|b_{+-lj}(z)| \leq \frac{C}{|\tilde{r}_l(z)|}, \quad |b_{-+lj}(z)| \leq \frac{C}{|\tilde{r}_l(z)|}.$$

*Proof.* First we show that in this region, for  $j \in \tilde{\mathcal{E}}$  and  $\chi \in L_{\text{comp}}^\infty(X)$ ,

$$(9) \quad \|e^{\mp i\tilde{r}_j(z)x}(I + VR_0(w_{\mathcal{E}}(z)))^{-1}\chi e^{\pm i\tilde{r}_j(z)x}\| \leq C$$

when  $|r_{j_0}(z)|$  is sufficiently large.

When  $|r_{j_0}(z)|$  is sufficiently large, and  $\tilde{\chi} \in L_{\text{comp}}^\infty(X)$  is one on the support of  $V$ ,

$$\begin{aligned} (10) \quad & \|\tilde{\chi} e^{\mp i\tilde{r}_j(z)x}(I + VR_0(w_{\mathcal{E}}(z)))^{-1}\chi e^{\pm i\tilde{r}_j(z)x}\| \\ &= \left\| \sum_{m=0}^{\infty} e^{\mp i\tilde{r}_j(z)x} (-1)^m (VR_0(w_{\mathcal{E}}(z))\tilde{\chi})^m \chi e^{\pm i\tilde{r}_j(z)x} \right\| \\ &= \left\| \sum_{m=0}^{\infty} (-1)^m (e^{\mp i\tilde{r}_j(z)x} VR_0(w_{\mathcal{E}}(z))\tilde{\chi} e^{\pm i\tilde{r}_j(z)x})^m \chi \right\| \\ &\leq C \end{aligned}$$

where we are using Lemma 3.1. Using this estimate and the definition of  $b_{+-lj}$ ,  $b_{-+lj}$ , we obtain the desired estimates. □

We shall need the following bound on the  $b_{++lj}(z)$  and  $b_{--lj}(z)$ .

**Lemma 3.3.** *Fix a sheet of  $\hat{Z}$  with corresponding labeling set  $\mathcal{E}$ , and let  $j_0 \in \mathcal{E}$ . If  $\operatorname{Im} r_{j_0}(z) \leq -\alpha < 0$ ,  $l, j \in \tilde{\mathcal{E}}$ , and  $\operatorname{supp}(V) \subset [-\beta, \gamma]$ , then for  $|r_{j_0}(z)|$  sufficiently large (depending on  $\alpha$ ),*

$$|b_{++lj}(z)| \leq \frac{Ce^{2\gamma|\operatorname{Im} \tilde{r}_j(z)|}}{|\tilde{r}_l(z)|}, \quad |b_{--lj}(z)| \leq \frac{Ce^{2\beta|\operatorname{Im} \tilde{r}_j(z)|}}{|\tilde{r}_l(z)|}.$$

*Proof.* We give the proof for  $b_{++lj}$ . Note that if  $\operatorname{supp} f \subset \operatorname{supp} V$ , then  $\operatorname{supp}(I + VR_0(w_{\mathcal{E}}(z)))^{-1}f \subset \operatorname{supp} V$ . Recall that  $\Phi_{j\pm}(x, y, z) = e^{\pm i\tilde{r}_j(z)x}\phi_j(y)$ . Then as in (10), we obtain that

$$\|\varphi_{j+}\| = \|(I + VR_0(w_{\mathcal{E}}(z)))^{-1}V\Phi_{j+}\| \leq Ce^{\gamma|\operatorname{Im} \tilde{r}_j(z)|}.$$

Using this bound and the remark about the support properties of  $(I + VR_0(w_{\mathcal{E}}(z)))^{-1}$ ,

$$\begin{aligned} |b_{++lj}(z)| &= \left| \frac{1}{2\tilde{r}_l(z)} \int_X \varphi_{j+}(x, y, z) \chi(x, y) \Psi_{l+}(x, y, z) d\operatorname{vol}_X \right| \\ &\leq \frac{C}{|\tilde{r}_l(z)|} e^{2\gamma|\operatorname{Im} \tilde{r}_j(z)|} \end{aligned}$$

for  $|r_{j_0}(z)|$  sufficiently large,  $\operatorname{Im} r_{j_0}(z) \leq -\alpha$ . For the last inequality we have also used that  $\tilde{r}_i(z) \rightarrow \tilde{r}_j(z)$  as  $\Pi(z) \rightarrow \infty$ .

A similar argument yields the proof of the bound for  $b_{--lj}(z)$ .  $\square$

*Proof of Theorem 1.1.* We use the coordinate  $k = r_{j_0}(z)$  to identify our fixed sheet with the lower half plane. Let  $g_1(k) = \det(I + B(z(k)))$ , where  $\Pi(z(k)) = k^2 + \nu_{j_0}^2$  and  $z$  lies on our sheet. Here  $B(z)$  is as defined in (4) and (5). Then  $g_1(k)$  has at most a finite number of poles,  $k_1, k_2, \dots, k_{m_\alpha}$ , listed with multiplicity, in  $\operatorname{Im} k \leq -\alpha$ . Let

$$g_2(k) = g_1(k)(k - k_1)(k - k_2) \cdots (k - k_{m_\alpha})$$

and, if  $g_2(-i\alpha) \neq 0$ , let

$$g_3(k) = \frac{g_2(k)}{g_2(-i\alpha)}.$$

If  $g_2(-i\alpha) = 0$ , let

$$g_3(k) = \frac{g_2(k)l!}{(k + i\alpha)^l g_2^{(l)}(-i\alpha)}$$

where  $l$  is chosen so that  $g_2^{(m)}(-i\alpha) = 0$  if  $m < l$  but  $g_2^{(l)}(-i\alpha) \neq 0$ . Then Lemmas 3.2 and 3.3 show that the hypotheses of Theorem 2.1 are satisfied for  $g_4(k) = g_3(k - i\alpha)$ , with

$$|h_{g_4}(\varphi)| \leq 2 \left( \sup_{(x,y), (x',y') \in \operatorname{supp} V} |x - x'| \right) \operatorname{card}(\tilde{\mathcal{E}}) |\sin \varphi|.$$

Recalling that, except possibly for a finite number, the zeros of  $g_3(k)$  correspond to the poles of  $R_V(z)$  in this region, an application of Theorem 2.1 finishes the proof.  $\square$

#### 4. DETERMINING $c_{V,\mathcal{E}}$ AND A COUNTEREXAMPLE

In this section we prove Theorem 1.2, give a counterexample, and give another example of a setting in which  $c_{V,\mathcal{E}}$  can be determined.

We shall need the following lemma, which is Lemma 4.1 of [10].

**Lemma 4.1.** *Suppose  $v \in L^\infty(\mathbb{R})$  has compact support contained in  $[-1, 1]$ , but in no smaller interval. Suppose  $f(x, k)$  is analytic for  $k$  in the lower half plane, and for real  $k$  we have  $f(x, k) \in L^2([-1, 1]dx, \mathbb{R}dk)$ . Then  $\int e^{\pm ikx} v(x)(1 - f(x, k))dx$  has exponential type at least 1 for  $k$  in the lower half plane.*

In the next lemma, we use  $k = \tilde{r}_l(z)$  as a coordinate, and, fixing a sheet of  $\hat{Z}$ , let  $z(k)$  be the corresponding point on  $\hat{Z}$ .

**Lemma 4.2.** *Let  $X = (-\infty, \infty) \times Y$  and suppose that the support of  $V$  is contained in  $[-b, b] \times Y$  and the interval  $[-b, b]$  cannot be replaced by a smaller one. Suppose that*

$$C|V_{ll}(x)| = C \left| \int_Y V(x, y) |\phi_l(y)|^2 d\text{vol}_Y \right| \geq |V(x, y)|, \text{ for } |x - b| < \epsilon, |x + b| < \epsilon$$

for some  $C, \epsilon > 0$ . Fix a sheet of  $\hat{Z}$  on which  $\text{Im } \tilde{r}_l(z) < 0$ , and choose  $\alpha$  so that there are no poles of  $b_{++ll}$ ,  $b_{--ll}$  on this sheet with  $\text{Im } r_l(z) \leq -\alpha$ . Then  $b_{++ll}(z(k))$ ,  $b_{--ll}(z(k))$  are functions of type at least  $2b$  for the half-plane  $\text{Im } k \leq -\alpha$ ,  $k = \tilde{r}_l(z)$ .

*Proof.* We give the proof for  $b_{++ll}$ , as the proof for  $b_{--ll}$  is similar.

Let  $g(k, x) = e^{ikx}$  and  $\mathcal{E}$  be the labeling set associated to our fixed sheet of  $\hat{Z}$ . Let

$$\begin{aligned} f_1(x, y, k) &= \overline{\phi}_l(y) \frac{1}{V(x, y)} \left[ \frac{1}{g} [I - [I + VR_0(\omega_{\mathcal{E}}(z(k)))]^{-1}] V \Phi_{l+}(\cdot, z(k)) \right] (x, y) \\ &= \overline{\phi}_l(y) \sum_{m=1}^{\infty} (-1)^m \left[ ((g)^{-1} R_0(\omega_{\mathcal{E}}(z(k))) V g)^m \phi_l \right] (x, y) \end{aligned}$$

where the second equality holds when  $|k|$  is sufficiently large. Then

$$b_{++ll}(z(k)) = \frac{i}{2k} \int e^{2ikx} V(x, y) (|\phi_l|^2(y) - f_1(x, y, k)) d\text{vol}_X.$$

Let

$$\chi_\epsilon(x) = \begin{cases} 0, & \text{if } |x| < b - \epsilon \text{ or } |x| > b \\ 1, & \text{if } b - \epsilon \leq |x| \leq b. \end{cases}$$

Let

$$v(x) = \int_Y V(x, y) |\phi_l|^2(y) d\text{vol}_Y = V_{ll}(x),$$

and

$$f(x, k) = \frac{1}{V_{ll}(x)} \chi_\epsilon(x) \int_Y V(x, y) f_1(k, x, y) d\text{vol}_Y.$$

Note that

$$b_{++ll}(z(k)) = \frac{i}{2k} \int e^{2ikx} v(x)(1-f(x,k))dx - \int_X e^{2ikx} (1-\chi_\epsilon) V(x,y) f_1(x,y,k) d\text{vol}_X.$$

Using (9) and the support properties of  $V(1-\chi_\epsilon)$ , the last term on the right is of type at most  $2b-2\epsilon$ , and so we need only show that the first integral on the right is of type at least  $2b$ . To do this, we will apply Lemma 4.1 to  $b_{++ll}(z(k+i\alpha))$ .

We must show that  $f(x,k) \in L^2([-b,b]dx, \mathbb{R}dk)$  when  $\text{Im } k = -\alpha$ . We have

$$\begin{aligned} \int |f(x,k)|^2 dx &= \int_{|x| \leq b} |V_{ll}(x)|^{-2} \chi_\epsilon(x) \left| \int_Y V(x,y) f_1(k,x,y) d\text{vol}_Y \right|^2 dx \\ &\leq \int_{|x| \leq b} \int_Y |V_{ll}(x)|^{-2} \chi_\epsilon(x) |V(x,y)|^2 d\text{vol}_Y \int_Y |f_1(k,x,y)|^2 d\text{vol}_Y dx \\ &\leq C \int_X |f_1(k,x,y)|^2 d\text{vol}_X. \end{aligned}$$

By Lemma 3.1, when  $|\text{Re } k|$  is sufficiently large, this is bounded by  $C|\text{Re } k|^{-7/6}$ . When  $|\text{Re } k|$  is in a compact set (with  $\text{Im } k = -\alpha$ ), it is enough to note that  $\int |f_1(k,x,y)|^2 d\text{vol}_X$  is bounded, so that  $f(x,k) \in L^2([-b,b]dx, \mathbb{R}dk)$ . Then, applying Lemma 4.1 after appropriately rescaling, we finish the proof.  $\square$

*Proof of Theorem 1.2.* We use  $k = r_{j_0}(z) = \tilde{r}_{l_0}(z)$  as the coordinate. The simplicity of  $\nu_{j_0}^2$  as an eigenvalue of  $\Delta_Y$  means that the matrix  $B$  is a  $2 \times 2$  matrix

$$B = \begin{pmatrix} b_{+-l_0l_0} & b_{--l_0l_0} \\ b_{++l_0l_0} & b_{-+l_0l_0} \end{pmatrix}.$$

Thus  $\det(I+B)(z(k)) = [(1+b_{+-l_0l_0})(1+b_{-+l_0l_0}) - b_{--l_0l_0}b_{++l_0l_0}](z(k)) = \varphi_1(k)$ .

Suppose first that  $\varphi_1(k)$  has no poles in the region  $\text{Im } k \leq -\alpha$ . If  $\varphi_1(-i\alpha) \neq 0$ , let

$$\varphi_2(k) = \frac{\varphi_1(k)}{\varphi_1(-i\alpha)}.$$

If  $\varphi_1(-i\alpha) = 0$ , let

$$\varphi_2(k) = \frac{\varphi_1(k)!}{(k+i\alpha)^l \varphi_1^{(l)}(-i\alpha)}$$

where  $l$  is chosen so that  $\varphi_1^{(m)}(-i\alpha) = 0$  if  $m < l$  but  $\varphi_1^{(l)}(-i\alpha) \neq 0$ .

Note that by Lemmas 3.2 and 3.3, for  $s \in \mathbb{R}$ ,  $\varphi_2(s-i\alpha) = c_0(1 + O(|s|^{-1}))$  when  $|s| \rightarrow \infty$ , for some nonzero constant  $c_0$ . Moreover, by Lemmas 3.2, 3.3, and 4.2,  $\varphi_2(k)$  is a function of type  $4b$  in the half plane  $\text{Im } k \leq -\alpha$ . Then applying Theorem 2.1 to  $\varphi_2(k)$  in the half-plane  $\text{Im } k \leq -\alpha$ , we obtain the result.

If  $\varphi_1(k)$  has poles in the region  $\text{Im } k \leq -\alpha$ , they can be handled in the same manner as in the proof of Theorem 1.1.  $\square$

We give a counterexample for Theorem 1.2. Let  $X = \mathbb{R} \times \mathbb{S}^1$ , and let  $V(x,y) = V_1(x)e^{imy}$  with  $V_1(x) \in L_{\text{comp}}^\infty(\mathbb{R})$  nontrivial and  $m > 0$  an integer. Then on the sheet with  $\text{Im } r_j(z) < 0$  if and only if  $j = 0$  there are no resonances. To see this,

restrict  $z$  to this sheet of  $\hat{Z}$  and note that since for integers  $n$   $R_0$  commutes with projection onto the span of  $e^{iny}$ ,

$$(VR_0(\omega_{\{0\}}(x)))^j V \Phi_{0,\pm} = u_{\pm,j}(x) e^{im(j+1)y}$$

for  $j = 0, 1, 2, \dots$ . Moreover,

$$(I + VR_0(\omega_{\{0\}}(z)))^{-1} V = \sum_{j=0}^{\infty} (-1)^j (VR_0(\omega_{\{0\}}(z)))^j V$$

when  $-\operatorname{Im} r_0(z) \geq \alpha > 0$  and  $|r_0(z)|$  is sufficiently large. Therefore, using (5), we see that  $b_{\pm+00}(z) = 0 = b_{\pm-00}(z)$  when  $-\operatorname{Im} r_0(z) \geq \alpha > 0$  and  $|r_0(z)|$  is sufficiently large. By analytic continuation,  $b_{\pm+00}(z) = 0 = b_{\pm-00}(z)$  for all  $z \in \hat{Z}$ . Thus, by the discussion of Section 3, there are no resonances on the sheet with  $\operatorname{Im} r_j(z) < 0$  if and only if  $j = 0$ .

In Theorem 1.2 we used some knowledge of the potential near the boundary of its support to allow us to find  $c_{V,\mathcal{E}}$ . In the following theorem we again make use of the fact that the potential is “controlled” near the boundary of its support.

**Theorem 4.1.** *Suppose for some potential  $V_0 \in L_{\text{comp}}^\infty(X; \mathbb{C})$ , with  $\operatorname{supp} V_0 \subset [-b_0, b_0] \times Y$  and for some sheet of  $\hat{Z}$  with corresponding labeling set  $\mathcal{E} \ni j_0$ , we have*

$\#\{z_k : z_k \text{ is a pole of } R_{V_0}(z) \text{ on this sheet},$

$$|r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\} = \frac{4b_0}{\pi} \#\{l : l \in \tilde{\mathcal{E}}\} r + o_\alpha(r)$$

for some  $\alpha > 0$ . Suppose in addition  $W \in L_{\text{comp}}^\infty(X; \mathbb{C})$  with  $\operatorname{supp} W \subset [-b_0 + \epsilon, b_0 - \epsilon] \times Y$  for some  $\epsilon > 0$ . Then

$\#\{z_k : z_k \text{ is a pole of } R_{V_0+W}(z) \text{ on this sheet},$

$$|r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\} = \frac{4b_0}{\pi} \#\{l : l \in \tilde{\mathcal{E}}\} r + o_\alpha(r).$$

That is, if the resonance-counting function for  $\Delta + V_0$  has maximal growth rate, so does that for  $\Delta + V_0 + W$ .

*Proof.* In the proof of this theorem, we will add a superscript to the matrix  $B$  from Section 3 and its entries to indicate to which potential it is associated. That is, when  $|r_{j_0}(z)|$  is sufficiently large, the poles of the resolvent of  $\Delta + V$  correspond to the zeros of  $\det(I + B^V(z))$  and likewise for  $V_0$ .

In this proof, as previously, we shall sometimes use as coordinate on our sheet  $k = r_{j_0}(z)$ , and then  $z(k)$  is the corresponding point on our sheet.

Let  $V = V_0 + W$ . We shall show that  $B^V(z) = B^{V_0}(z) + D(z)$ , with the entries  $d_{lj}(z)$  of  $D(z)$  satisfying

$$(11) \quad |d_{lj}(z)| \leq \frac{C}{|\tilde{r}_l(z)|} e^{(2b_0 - \epsilon)|\operatorname{Im} \tilde{r}_j(z)|}.$$

Because of the assumption on the distribution of resonances for  $\Delta + V_0$ ,  $\det(I + B^{V_0}(z(k)))$  is of type  $4b_0\#\{l : l \in \tilde{\mathcal{E}}\}$  in  $\operatorname{Im} k < -\alpha < 0$ . Moreover, each entry of  $B^{V_0}(z(k))$  has type at most  $2b_0$  and is bounded by  $Ce^{2b_0|\operatorname{Im} k|}$ . Then

$$\begin{aligned}\det(I + B^V(z(k))) &= \det(I + B^{V_0}(z(k)) + D(z(k))) \\ &= \det(I + B^{V_0}(z(k))) + O\left(\frac{e^{|\operatorname{Im} k|(4b_0\#\{l:l\in\mathcal{E}\}-\epsilon)}}{|k|}\right).\end{aligned}$$

Applying Theorem 2.1 as in the proof of Theorems 1.1 and 1.2 finishes the proof.

It remains to show (11). Note that we may write, when  $|r_{j_0}(z)|$  is sufficiently large,

$$\begin{aligned}[I + (V_0 + W)R_0(\omega_{\mathcal{E}}(z))]^{-1} &= \\ \left(I + \sum_{m=1}^{\infty} (-1)^m \left[(I + V_0 R_0(\omega_{\mathcal{E}}(z)))^{-1} W R_0(\omega_{\mathcal{E}}(z))\right]^m\right) [I + V_0 R_0(\omega_{\mathcal{E}}(z))]^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\varphi_{l+}^V(z) &= (I + V R_0(\omega_{\mathcal{E}}(z)))^{-1} (V_0 + W)\Phi_{l+}(\bullet, z) \\ &= (I + V_0 R_0(\omega_{\mathcal{E}}(z)))^{-1} (V_0\Phi_{l+}(\bullet, z)) + (I + V_0 R_0(\omega_{\mathcal{E}}(z)))^{-1} (W\Phi_{l\pm}(\bullet, z)) \\ &\quad + \sum_{m=1}^{\infty} (-1)^m \left[(I + V_0 R_0(\omega_{\mathcal{E}}(z)))^{-1} W R_0(\omega_{\mathcal{E}}(z))\right]^m [I + V_0 R_0(\omega_{\mathcal{E}}(z))]^{-1} \Phi_{l+}(\bullet, z).\end{aligned}$$

The first term on the right is  $\varphi_{l+}^{V_0}(z)$ . The second term is, as in (10), bounded by  $Ce^{(b_0-\epsilon)|\operatorname{Im} r_l(z)|}$ . Again as in (10), the third term is also bounded by  $Ce^{(b_0-\epsilon)|\operatorname{Im} r_l(z)|}$ . Putting all this into the definition of  $b_{++lj}^V(z)$ , we see that

$$b_{++lj}^V(z) = b_{++lj}^{V_0}(z) + O\left(\frac{e^{(2b_0-\epsilon)|\operatorname{Im} r_{j_0}(z)|}}{|r_{j_0}(z)|}\right).$$

A similar argument works for the other entries of  $B^V(z)$ , proving (11).  $\square$

Combining the previous theorem with the results for potential scattering in one dimension [10, 17], we obtain the following corollary.

**Corollary 4.1.** *Let  $V(x, y) = V_0(x) + W(x, y) \in L_{\text{comp}}^\infty(X; \mathbb{C})$ , where the support of  $V_0$  is contained in  $[-b, b]$  and in no smaller interval and  $\operatorname{supp} W \subset [-b + \epsilon, b - \epsilon] \times Y$  for some  $\epsilon > 0$ . Then on any sheet of  $\hat{Z}$  with corresponding labeling set  $\mathcal{E}$ ,*

*$\#\{z_k : z_k \text{ is a pole of the resolvent of } \Delta + V \text{ on this sheet},$*

$$|r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\} = \frac{4b}{\pi} \#\{l : l \in \tilde{\mathcal{E}}\}r + o_\alpha(r)$$

*for any  $\alpha > 0$ .*

## 5. RESULTS FOR HALF-CYLINDERS

In this section, we consider half-cylinders  $X = [0, \infty) \times Y$ , with  $\Delta$  either the Dirichlet or Neumann Laplacian on  $X$ . Let  $V \in L_{\text{comp}}^\infty(X; \mathbb{C})$ . The resolvent  $(\Delta + V - z)^{-1}$  has a meromorphic continuation to  $\hat{Z}$  just as in the full cylinder case. We give several results analogous to the results for full cylinders. Since the proofs are so similar, we only sketch them.

Let  $R_{0\pm}(z) = (\Delta - z)^{-1}$  be the resolvent for the Neumann (+) or Dirichlet (-) Laplacian on  $X$ , for  $z \in \hat{Z}$ . Restrict  $z$  to one fixed sheet of  $\hat{Z}$ , with corresponding labeling set  $\mathcal{E}$ . Then, following the same argument as in the beginning of Section 3, we can show that when  $|\text{Im } \Pi(\omega_{\mathcal{E}}(z))| > \|\text{Im } V\|_{L^\infty}$ , the poles of the resolvent of  $\Delta + V$  correspond to the zeros of  $\det(I + B_\pm(z))$ . Here we are again using “+” for the Neumann Laplacian and “-” for the Dirichlet Laplacian. To define  $B_\pm(z)$ , let

$$\begin{aligned}\Phi_{\pm l}(x, y, z) &= (e^{i\tilde{r}_l(z)x} \pm e^{-i\tilde{r}_l(z)x})\phi_l(y) \\ \varphi_{\pm l}(x, y, z) &= ((I + VR_{0\pm}(\omega_{\mathcal{E}}(z)))^{-1}(V\Phi_{\pm l})(\bullet, z))(x, y).\end{aligned}$$

Then  $B_\pm(z) = (b_{\pm jk}(z))_{jk \in \mathcal{E}}$ , with

$$b_{\pm jk}(z) = \frac{i}{2\tilde{r}_j(z)} \int_X \varphi_{\pm j}(x, y, z) \overline{\Phi}_{\pm k}(x, y, z) d\text{vol}_X.$$

We obtain the following analog of Theorem 1.1.

**Theorem 5.1.** *Let  $X = [0, \infty) \times Y$  and let  $V \in L_{\text{comp}}^\infty(X; \mathbb{C})$ , with  $\text{supp } V \subset [0, b] \times Y$ . Fix a sheet of  $\hat{Z}$ , and suppose that  $\text{Im } r_{j_0}(z) < 0$  on this sheet. Then, there is a constant  $c_{V,\mathcal{E}} \geq 0$  such that for any  $\alpha > 0$ ,*

$$\begin{aligned}\#\{z_k : z_k \text{ is a pole of the resolvent on this sheet}, |r_{j_0}(z_k)| < r, \text{Im } r_{j_0}(z_k) < -\alpha\} \\ = c_{V,\mathcal{E}}r + o_\alpha(r)\end{aligned}$$

The constant  $c_{V,\mathcal{E}}$  depends on the potential  $V$  and the sheet. Moreover,

$$c_{V,\mathcal{E}} \leq \frac{2b}{\pi} \#\{l : \text{Im } \tilde{r}_l(z) < 0 \text{ when } z \text{ lies on this sheet}\}.$$

*Proof.* Just as in the proof of Lemmas 3.2 and 3.3, we can show that on our fixed sheet

$$|b_{\pm jk}(z)| \leq \frac{C}{|\tilde{r}_j(z)|} e^{2b|\text{Im } \tilde{r}_k(z)|}.$$

Then the proof follows just as the proof of Theorem 1.1.  $\square$

**Theorem 5.2.** *Let  $X = [0, \infty) \times Y$  and suppose that the support of  $V$  is contained in  $[0, b] \times Y$  and the number  $b$  cannot be replaced by a smaller one. Restrict ourselves to a sheet of  $\hat{Z}$  with  $\text{Im } r_j(z) < 0$  if and only if  $j = j_0$ . Suppose that  $\nu_{j_0}^2$  is a simple eigenvalue of  $\Delta_Y$ , with  $\nu_{j_0}^2 = \sigma_{l_0}^2$ . Suppose, in addition, that*

$$C|V_{l_0 l_0}(x)| = C \left| \int_Y V(x, y) |\phi_{l_0}(y)|^2 d\text{vol}_Y \right| \geq |V(x, y)|, \text{ for } |x - b| < \epsilon$$

for some  $C, \epsilon > 0$ . Then, for any  $\alpha > 0$ ,

$$\begin{aligned} \#\{z_k : z_k \text{ is a pole of the resolvent on this sheet}, |r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\} \\ = \frac{2}{\pi} br + o_\alpha(r). \end{aligned}$$

*Proof.* In this case  $B_\pm(z)$  is a single function,  $b_{\pm l_0 l_0}$ . Let  $k = \tilde{r}_{l_0}(z)$  and let  $z(k)$  be the corresponding point on  $\hat{Z}$ . We have

$$\begin{aligned} b_{\pm l_0 l_0}(z(k)) &= \frac{i}{2k} \int_X (e^{ikx} \pm e^{-ikx}) \bar{\phi}_{l_0} [[I + VR_0(w_\varepsilon(z(k)))]^{-1} V \Phi_{l\pm}(\bullet, z(k))] d\operatorname{vol}_X \\ &= \frac{i}{2k} \int_X e^{ikx} \bar{\phi}_{l_0} [I + VR_0(w_\varepsilon(z(k)))]^{-1} V f_{l_0}(\bullet, z(k)) d\operatorname{vol}_X + O(e^{| \operatorname{Im} k |}). \end{aligned}$$

Here  $f_{l_0}(x, y, z) = e^{i\tilde{r}_{l_0}(z)x} \phi_{l_0}(y)$ , and we have used a bound similar to that of Lemma 3.1 to obtain the bound  $O(e^{| \operatorname{Im} k |})$  on the rest. Following the technique of Lemmas 3.2 and 4.2 shows that  $b_{\pm l_0 l_0}(z(k))$  is an exponential function of type 2b for  $\operatorname{Im} k \leq -\alpha$ . The proof is completed as in the proof of Theorem 1.2.  $\square$

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